XV. *Of the construction of Logarithmic Tables.* *By* Thomas Knight, *Esq.* *Communicated by* Taylor Combe, *Esq. Sec. R. S.*

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1. I HAVE endeavoured, in this short Paper, to give a *simple* and *connected* theory of the construction of logarithms, which I think has not hitherto been done.

Prop. I.

To find the Logarithm of $1 + x$.*

It is not difficult to see that we may assume

$L(1+x) = {'A}x + {''A}x^2 + {'''A}x^3 + {''''A}x^4 + \&c.$, whence

$L(1+y) = {'A}y + {''A}y^2 + {'''A}y^3 + {''''A}y^4 + \&c.$, and

$L\{(1+x)(1+y)\} = L(1+x+y+xy)$, or putting $1+x=\pi$,

$= L\{1+(x+\pi y)\} = {'A}(x+\pi y) + {''A}(x+\pi y)^2 + {'''A}(x+\pi y)^3 + \&c.$

If we substitute these three expansions in the equation

$$L(1+x) + L(1+y) = L\{(1+x)(1+y)\}$$

which expresses the nature of logarithms, and compare the coefficients of the first power of y, we find

$$'A = {'A}\pi + 2{''A}\pi x + 3{'''A}\pi x^2 + 4{''''A}\pi x^3 + \&c.$$

or

$$\begin{aligned} 'A = {'A} &+ 2{''A}x + 3{'''A}x^2 + 4{''''A}x^3 + \&c. \\ &+ {'A}\big| \quad + 2{''A}\big| \quad + 3{'''A}\big| \end{aligned}$$

whence, by comparing the coefficients of the powers of x,

${'A}={'A}$, $2{''A}+{'A}=0$, $3{'''A}+2{''A}=0$, $4{''''A}+3{'''A}=0$, &c.; or

${'A}={'A}$, ${''A} = -\frac{'A}{2}$, ${'''A} = -\frac{2''A}{3} = \frac{'A}{3}$, ${''''A} = -\frac{3A'''}{4} = -\frac{'A}{4}$, &c.

* I find that the method of expansion made use of in this Proposition had been previously employed by Mr. Spence.

and $$L(1+x, ='A\left\{\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\right\}$$

As for 'A it may be evidently taken at pleasure; and innumerable systems of logarithms may be formed by assigning different values to it, for

$$'A\,L(1+x)+'AL(1+y)='AL\{(1+x)(1+y)\}$$

which expresses that, if every logarithm in a system be multiplied by the same constant quantity, the products will still form a system of logarithms to the same numbers.

Cor. By an easy transformation of $L(1+x)$, we get for BRIGG's logarithms, M being the modulus,

$L.\frac{a}{b}=2M\left\{\frac{a-b}{a+b}+\frac{1}{3}\left(\frac{a-b}{a+b}\right)^3+\frac{1}{5}\left(\frac{a-b}{a+b}\right)^5+\right\}$; and whenever the logarithm of a fraction is spoken of in the following proposition, it is supposed to be found by this series.

2. How are we to begin, in forming a table of logarithms?

DELAMBRE (Preface to BORDA, p. 75) says, that we should begin at 10,000; and the same writer (Mémoires de l'Institut, Tome cinq. p. 65), speaking of the great French Tables, says that the logarithms of primes under 10,000 were calculated directly by series, and those of numbers above 10,000 by six orders of differences.

Now it is not easy to see, why any of the logarithms in the lower half of the Table, except those of the numbers 2 and 3, should be computed directly; since they may be got, each by a single subtraction, from those in the upper half. Suppose, for instance, there had been found directly the logarithms of numbers from 100,000 to 200,000; those of numbers down to 50,000 are found by merely subtracting the logarithm of 2, successively, from those of all the even numbers;

beginning at the top of the Table, with L. 1999998, L. 1999996, &c., and setting down the remainder for the logarithms of the successive numbers below 100,000, viz. L. 99999, L. 99998, &c.

When we have got down to 50,000, if we were to proceed in the same way, we should have to operate on the logarithms thus obtained, between 100,000 and 50,000: If, therefore, we fear any accumulation of errors, we may (because $3 \times 49999 = 149997$) subtract the logarithm of 3 from L. 149997, and from the logarithm of every third number going downward, and set the remainders down successively for the logarithms of numbers below 50,000. And thus we may proceed till we get somewhat below 34,000; then the logarithm of 4 will carry us down to 25,000; and the logarithm of 5 to 20,000, which completes the work, those below 20,000 having been already found.

In the great French Tables, however, it has been thought proper to calculate the logarithms of numbers under 10,000 with more decimal places than the rest. These must necessarily be found independently of the others; as they form in reality a separate Table.

In the next proposition, is contained a general method of finding converging series for the calculation of logarithms. The propositions which follow this are only corollaries from it, and give forms for interpolation; so that *every thing* relating to the construction of logarithms is effected by one simple and uniform process.

PROP. II.

3. *To express a number* (x) *by the product of a series of fractions converging continually towards unity.*

Let n, n', n'', &c. be numbers much less than x; in the equation $x = x$, change x, in the second member, into $x + n$, and multiply by such a factor as will restore the equality; there arises $x = (x + n) \times \frac{x}{x+n}$. If, in the second member of this equation, we change x into $x + n'$, in the *last factor* $\frac{x}{x+n}$, and multiply by such a fractional factor as shall again restore the equality, we have

$$x = (x + n) \times \frac{x+n'}{x+n+n'} \times \frac{x(x+n+n')}{(x+n)(x+n')}.$$

If here, in like manner, we change x into $x + n''$ in the last factor, and restore the equality as before, by annexing a new factor, then

$$x = (x+n) \times \frac{x+n'}{x+n+n'} \times \frac{(x+n'')(x+n+n'+n'')}{(x+n+n'')(x+n'+n'')} \times \frac{x(x+n+n')(x+n+n'')(x+n'+n'')}{(x+n)(x+n')(x+n'')(x+n+n'+n'')}$$

and the same process may be repeated as long as we think it necessary. Now it is plain that the last annexed factor, as we continue these operations, must always approach nearer to unity than that which was the last before; thus, n being very small compared with x, $\frac{x}{x+n}$ does not much differ from unity, and when $x + n'$ is put for x in this fraction (n' being also very small compared with x) its value will be nearly the same as before: of course the annexed factor $\frac{x(x+n+n')}{(x+n)(x+n')}$ will differ very little from unity: and it will differ from it much less than the preceding factor $\frac{x}{x+n}$; for let $\frac{x}{x+n}=1-\mu$, $\frac{x+n'}{x+n+n'}=1-\mu'$, and μ and μ' being small fractions; the new factor $\frac{x(x+n+n')}{(x+n)(x+n')}=\frac{1-\mu}{1-\mu'}=1-(\mu-\mu')$ nearly: and consequently differs less from unity than the factor which was last before.

4. These equations, being put into logarithms, give a series of converging expressions for the logarithm of x. We have successively,

1st. $L.x=L(x+n)+L\left(\frac{x}{x+n}\right)$

2d. $L.x=L(x+n)+L(x+n')-L(x+n+n')+L\left(\frac{x^2+(n+n')x}{x^2+(n+n')x+nn'}\right)$;

but before we put the third equation into logarithms, it will be better to simplify it; one of the most obvious ways of doing which is to make $n+n'=n''$; then

3d. $L.x=L(x+n)+L(x+n')+L(x+2n'')-L(x+n+n'')-L(x+n'+n'')+L\left(\frac{x^3+3n''x^2+(nn'+2n''^2)x}{x^3+3n''x^2+(nn'+2n''^2)x}\right)$.

This may be still farther simplified by making $n=n'$, consequently $n''=2n$, then

$$L.x=2L.(x+n)-2L(x+3n)+L(x+4n)+L\left(\frac{x^3+6nx^2+9n^2x}{x^3+6nx^2+9n^2x+4n^3}\right)$$

If now we change n into -1, and x into $x+2$, we shall fall upon the elegant formula of Mr. BORDA.*

* If any one shall attempt to calculate a Table of logarithms by means of differences, each logarithm being got, not from the next, but from the next but one below it, he will fall upon the series of BORDA: for $\Delta^3.L(x-1)=L.(x+2)-3L(x+1)+3L.x-L(x-1)$; $\Delta^3.L(x-2)=L(x+1)-3L.x+3L(x-1)-L(x-2)$, by adding which, $\Delta^3.L(x-1)+\Delta^3.L(x-2)=L\left(\frac{(x+2)(x-1)^2}{(x-2)(x+1)^2}\right)$ which gives the series we are speaking of. This remark will be exemplified in one of the following Propositions.

In like manner we might investigate approximations of the fourth and following orders: but this kind of research has very little use, and the Proposition was inserted for a different purpose.

PROP. III.

5. *Supposing that, in the last Proposition,* $n = n' = n'' = \&c. = -1$. *It is required to find the law of the converging expressions for* L . x.

In this case the four first transformations give

$$x = (x-1) \times \frac{x}{x-1}$$

$$x = (x-1) \times \frac{x-1}{x-2} \times \frac{x(x-2)}{(x-1)^2},$$

$$x = (x-1) \times \frac{x-1}{x-2} \times \frac{(x-1)(x-3)}{(x-2)^2} \times \frac{x(x-2)^3}{(x-1)^3(x-3)}$$

$$x = (x-1) \times \frac{x-1}{x-2} \times \frac{(x-1\,(x-3)}{(x-2)^2} \times \frac{(x-1)(x-3)^3}{(x-2)^3(x-4)} \times \frac{x(x-2)^6(x-4)}{(x-1)^4(x-3)^4}$$

which, put into logarithms, give

$$L\,.\,x = L(x-1) + L\left(\frac{x}{x-1}\right)$$

$$L\,.\,x = 2L(x-1) - L(x-2) + L\left(\frac{x(x-2)}{(x-1)^2}\right)$$

$$L\,.\,x = 3L(x-1) - 3L(x-2) + L(x-3) + L\left(\frac{x(x-2)^3}{(x-1)^3(x-3)}\right)$$

$$L\,.\,x = 4L(x-1) - 6L(x-2) + 4L(x-3) - L(x-4) + L\left(\frac{x(x-2)^6(x-4)}{(x-1)^4(x-3)^4}\right);$$

where a coincidence may be observed between the coefficients and those in the binomial theorem; and it is easily shown that the same coincidence will have place, how far soever we continue the method; or that, in general, the converging expression will be

$$L\,.\,x = \frac{n}{1}L(x-1) - \frac{n(n-1)}{1.2}L(x-2) + \frac{n(n-1)(n-2)}{1.2.3}L(x-3) - \ldots$$

$$\ldots\ldots\ldots + L\left\{\frac{x \times (x-2)^{\frac{n(n-1)}{1.2}} \times (x-4)^{\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}} + \&c.}{(x-1)^{\frac{n}{1}} \times (x-3)^{\frac{n(n-1)(n-2)}{1.2.3}} \times \&c.}\right\} (a)$$

For, if it be denied, let this represent a single result, to pass on to the next, we change x into $x-1$, in the logarithm of the fraction, and add a new logarithm (L) to restore the equality: the equation will thus become

$$L.x = \left.\begin{matrix}\frac{n}{1}\\ +1\end{matrix}\right\} L(x-1) \left.\begin{matrix}-\frac{n(n-1)}{1.2}\\ -\frac{n}{1}\end{matrix}\right\} L(x-2) \left.\begin{matrix}+\frac{n(n-1)(n-2)}{1.2.3}\\ +\frac{n(n-1)}{1.2}\end{matrix}\right\} L(x-3) - \ldots + (L)$$

or $L.x = \frac{n+1}{1} L(x-1) - \frac{(n+1)n}{1.2} L(x-2) + \frac{(n+1)n(n-1)}{1.2.3} L(x-3) - \ldots + (L)$,

and, by transposition, we find

$$(L) = L\left\{\frac{x \times (x-2)^{\frac{(n+1)n}{1.2}} \times (x-4)^{\frac{(n+1)n(n-1)(n-2)}{1.2.3.4}} \times \&c.}{(x-1)^{\frac{n+1}{1}} \times (x-3)^{\frac{(n+1)n(n-1)}{1.2.3}} \times \&c.}\right\}$$

so that the whole expression is of the same form as before, which is therefore proved to be general.

6. *Cor.* 1. If in the values of x. in the last article, we put for x, in the second $x+1$, in the third $x+2$, in the fourth $x+3$, and so on; and moreover represent the last fractions arising after such substitution by $\alpha, \alpha', \alpha'', \alpha'''$, &c., we get the following set of equations

$$\begin{aligned} x &= (x-1) \times \alpha \\ x+1 &= x \times \alpha \times \alpha' \\ x+2 &= (x+1) \times \frac{x+1}{x} \times \alpha' \times \alpha'' \qquad (b) \\ x+3 &= (x+2) \times \frac{x+2}{x+1} \times \frac{x(x+2)}{(x+1)^2} \times \alpha'' \times \alpha''' \\ &\&c. \ldots\ldots\ldots\ldots \&c. \ldots\ldots \end{aligned}$$

which, *from the manner of their formation*, are subject to this law, that the m^{th} fraction (provided it is not the last) in the value of $x+n$, is equal to $x+n-1$ divided by the product of the first $m-1$ fractions in the expression of $x+n-1$.

If, for brevity, we put $\underset{1}{L}$, L°, L', L'', &c. for $L(x-1)$, $L.x$, $L(x+1$, $L(x+2)$, &c. the last equations give

$$L^{\circ} = \underset{1}{L} + L.\alpha$$

$$L' = L^{\circ} + L.\alpha + L\alpha' \qquad (c)$$

$$L'' = L' + (L' - L^{\circ}) + L.\alpha' + L.\alpha''$$

$$L''' = L'' + (L'' - L') + (L'' - 2L' + L^{\circ}) + L.\alpha'' + L.\alpha'''$$

$$L'''' = L''' + (L''' - L'') + (L''' - 2L'' + L') + (L''' - 3L'' + 3L' - L^{\circ}) + L.\alpha''' + L.\alpha''''$$

. . . .

$$L^{''\cdots\cdots(n+1)} = L^{''\cdots n} + (L^{''\cdots n} - L^{''\cdots\cdots(n-1)}) + (L^{''\cdots n} - 2L^{''\cdots\cdots(n-1)} + L^{''\cdots(n-2)}) + \cdots\cdots + (L^{''\cdots n} - nL^{''\cdots(n-1)} + \frac{n(n-1)}{1.2}L^{''\cdots(n-2)} - \frac{n(n-1)(n-2)}{1.2.3}L^{''\cdots\cdots(n-3)} + \,) + L.\alpha^{''\cdots n} + L.\alpha^{''\cdots(n+1)}$$

These equations are subject to a law arising from that which we noticed in (*b*), viz. that the m^{th} term (provided it is not the last) in the value of $L^{''\cdots n}$ is equal to $L^{''\cdots(n-1)}$, minus the sum of the first $m-1$ terms in the expression $L^{''\cdots(n-1)}$. By *term* I here mean the whole expression included between two brackets.

If we form $L\alpha^{''\cdots\cdots n}$ generally from the last term of equation (*a*) we have

$$L.\alpha^{''\cdots n} = L\left\{\frac{(x+n)\times(x+n-2)^{\frac{(n+1)n}{1.2}}\times(x+n-4)^{\frac{(n+1)n(n-1)(n-2)}{1.2\,3\,4}}\times\&c.}{(x+n-1)^{\frac{n+1}{1}}\times(x+n-3)^{\frac{(n+1)n(n-1)}{1.2.3}}\times\&c.}\right\}$$

If any one should not be satisfied that the form given to $L^{''\cdots(n+1)}$ is general, he has only to form $L^{''\cdots\cdots(n+2)}$ from it, and he will find the same form in that case. Now $L^{''\cdots(n+2)}$ is formed from $L^{''\cdots\cdots(n+1)}$ by changing x into $x+1$ (or $L^{''\cdots\cdot r}$ into $L^{''\cdots(r+1)}$ in all the terms but the last $L.\alpha^{''\cdots(n+1)}$, and this we may remember had x first changed into $x-1$, and afterwards into $x+1$, so that it receives no alteration. Finally there is to be added $L.\alpha''^{\cdots(n+2)}$.

7. *Cor.* 2. Change x into $x-1$ in equations (b), and let the new values of α, α', α'', &c. be β, β', β'', &c., and there arises the following series of equations,

$$x-1=(x-2)\times\beta$$
$$x=(x-1)\times\beta\times\beta'$$
$$x+1=x\times\frac{x}{x-1}\times\beta'\times\beta''$$
$$x+2=(x+1)\times\frac{x+1}{x}\times\frac{(x-1)(x+1)}{x^2}\times\beta''\times\beta'''$$
$$\&c. \qquad\qquad \&c.$$

which being multiplied by equations (b), the first by the first, the second by the second, and so on, factor by factor, and putting $\alpha\times\beta=s$, $\alpha'\times\beta'=s'$, $\alpha''\times\beta''=s''$, and so on, give

$$x=(x-2)\times s$$
$$x+1=(x-1)\times s\times s'$$
$$x+2=x\times\frac{x+1}{x-1}\times s'\times s''$$
$$x+3=(x+1)\times\frac{x+2}{x}\times\frac{(x+2)(x-1)}{(x+1)x}\times s''\times s'''$$
$$\&c. \qquad\qquad \&c.$$

which, being put into logarithms, give

$$L^{\circ}=L_{''}+L.s$$
$$L'=L_{'}+L.s+L.s' \qquad (d)$$
$$L''=L^{\circ}+(L'-L_{'})+L.s'+L.s''$$
$$L'''=L'+(L''-L^{\circ})+(L''-L'-L^{\circ}+L_{'})+L.s''+L.s'''$$

. . . .

$$L''^{\cdots\cdots(n+1)}=L''^{\cdots\cdots(n-1)}+(L''^{\cdots\cdots n}-L''^{\cdots\cdots(n-2)})+(L''^{\cdots\cdots n}-L''^{\cdots(n-1)}$$

$-L''^{\ldots(n-2)}+L''^{\ldots(n-3)})+\ldots\ldots\ldots+(L''^{\ldots n}-(n-1)L''^{\ldots(n-1)}+\frac{n\times(n-3)}{1.2}L''^{\ldots(n-2)}-\frac{n(n-1)\times(n-5)}{1.2.3}L''^{\ldots(n-3)}+)+L.s''^{\ldots n}+$ $+L.s''^{\ldots(n+1)}$; any immediate term, as the $m+1^{th}$, will be of the form $L''^{\ldots n}-(m-1)L''^{\ldots(n-1)}+\frac{m\times(m-3)}{1.2}L'''^{(n-2)}-\frac{m(m-1)\times(m-5)}{1.2.3}+.$*

We easily see that

$$L.s''^{\ldots n}=L\left\{\frac{(x+n)\times(x+n-2)^{\frac{(n+1)\times(n-2)}{1.2}}\times(x+n-4)^{\frac{(n+1)n(n-1)\times(n-6)}{1.2.3.4}}\times\&c.}{(x+n-1)^{n}\times(x+n-3)^{\frac{(n+1)n\times(n-4)}{1.2.3}}\times\&c.}\right\}$$

It is scarcely necessary to say that $L.\alpha''^{\ldots n}$, $L.s''^{\ldots n}$ are to be expanded by the common series for $L.\frac{a}{b}$. viz.

* These forms are analogous to an expression in the method of differences, which, though not noticed by STIRLING and other writers on interpolation, may be useful on many occasions, as the coefficients are small and few in number. BORDA's expression for logarithms is a particular case of it.

$$u_n=(n-1)u_{n-1}-\frac{n\times(n-3)}{1.2}u_{n-2}+\frac{n(n-1)\times(n-5)}{1.2.3}n_{n-3}-\frac{n(n-1)(n-2)\times(n-7)}{1.2.3.4}u_{n-4}$$
$+\ldots\ldots\ldots+\Delta^n u+\Delta^n u_{-1}$. If we make $u_n=L.x$, we have, by taking n (in the coefficients) successively 1, 2, 3, 4, &c.

$L.x=L(x-2)+$ series,

$L.x=L(x-1)+L(x-2)-L(x-3)+$ series,

$L.x=2\{L(x-1)-L(x-3)\}+L(x-4)+$ series, (BORDA's if we change x into $x+2$).

$L.x=3\{L(x-1)+L(x-4)\}-2\{L(x-2)+L(x-3)\}-L(x-5)+$series,

$L.x=4\{L(x-1)-L(x-5)\}-5\{L(x-2)-L(x-4)\}+L(x-6)+$series,

$L.x=5\{L(x-1)+L(x-3)+L(x-4)+L(x-6)\}-9\{L(x-2)+L(x-5\}-L(x-7(+$series,

$L.x=6\{L(x-1)-L(x-7)\}-14\{L(x-2-L(x-3)+L(x-5)-L(x-6)\}+L(x-8)+$series,

&c. &c.

which may be useful, when taken without the series, as formulas of verification.

$L.\frac{a}{b}=2M\left\{\frac{a-b}{a+b}+\frac{1}{3}\left(\frac{a-b}{a+b}\right)^3+\frac{1}{5}\left(\frac{a-b}{a-b}\right)^5+\&c.\right\}$ though, for the purpose of the two following propositions, they will be better when expanded into series of monomials; see Prop. VI.

PROP. IV.

8. *To construct a Table of Logarithms by means of interpolation from the converging expressions* $L.\alpha$, $L.\alpha'$, $L.\alpha''$, &c.

When treating of the equations marked (c), we noticed a law to which the terms are subject; this law affords an easy method of eliminating the second, third, &c. terms, and, by this means, we find, successively,

$$\begin{aligned} L.x &= L(x-1)+L.\alpha \\ L(x+1) &= L.x+L.\alpha+L.\alpha' \qquad (e) \\ L(x+2) &= L(x+1)+L.\alpha+2L.\alpha'+L.\alpha'' \\ L(x+3) &= L(x+2)+L.\alpha+3L.\alpha'+3L.\alpha''+L.\alpha''' \\ &\dots\dots\dots\dots \\ L(x+n) &= L(x+n-1)+L.\alpha+nL.\alpha'+\tfrac{n(n-1)}{1.2}L.\alpha''+ \end{aligned}$$

If any one doubts whether this form is general, for every value of n, let it be only a single value; and supposing α, α', α'', &c. to become α_1, α'_1, α''_1, &c., by the substitution of $x+1$ for x, we have

$$L(x+n+1)=L(x+n)+L.\alpha_1+nL.\alpha'_1+\tfrac{n(n-1)}{1.2}L.\alpha''_1+$$

Now, if we consider the manner in which the last fractional factors, in the values of x, at the beginning of the last proposition, were formed from one another; and the change which they afterwards underwent in forming equations (b), we shall easily perceive that

$\alpha^{''\ldots n}=\frac{\alpha^{''\ldots(n-1)}}{\alpha^{''\ldots(n-1)}}$; whence $L.\alpha^{''\ldots(n-1)}=L.\alpha^{''\ldots n}+L.\alpha^{''(n-1)}$

by means of which equation, $L(x+n+1)$ becomes

$$L(x+n+1)=L(x+n)+L.\alpha+\frac{n}{1}L.\alpha'+\frac{n(n-1)}{1.2}L.\alpha''+\frac{n(n-1)(n-2)}{1.2.3}L\alpha'''+$$

$$+L.\alpha'+\frac{n}{1}L\alpha''+\frac{n(n-1)}{1.2}L.\alpha'''+$$

or $L(x+n+1)=L(x+n)+L.\alpha+\frac{n+1}{1}L.\alpha'+\frac{(n+1)n}{1.2}L.\alpha''+\frac{(n+1)n(n-1)}{1.2.3}L.\alpha'''+$ the same form as before, which is therefore general.

9. By means of equations (e) the logarithm of a number is found from the logarithm of that which precedes it, and the process will consist of a number of additions, equal to the number of the values of α we make use of.

Thus, if $L.\alpha'''$ be so small, as not to affect the last figure, to which we intend to carry the logarithms, it may be neglected and we must make

$$L.\alpha+L.\alpha'=r$$
$$L.\alpha+2L.\alpha'+L\,\alpha''=r+L.\alpha'+L.\alpha''=r'$$
$$L.\alpha+3L.\alpha'+3L.\alpha''=r'+L\alpha'+2L.\alpha''=r''$$
$$L.\alpha+4L.\alpha'+6L.\alpha''=r''+L.\alpha'+3L.\alpha''=r'''$$

. . . .

$$L\alpha+nL.\alpha'+\frac{n(n-1)}{1.2}L.\alpha''=r^{''\ldots(n-2)}+L.\alpha'+(n-1)L.\alpha''=r^{''\ldots(n-1)}$$

Here the quantities r', r'', r''', &c. are formed by two additions each, one more gives the logarithms; for, by substituting in (e),

$$L.x=L(x-1)+L.\alpha$$
$$L(x+1)=L.x+L\alpha+L\alpha'=L.x+r$$

$$L(x+2)=L(x+1)+r+L.\alpha'+L.\alpha''=L(x+1)+r'$$
$$L(x+3)=L(x+2)+r'+L.\alpha'+2L.\alpha''=L(x+2)+r''$$
$$L(x+4)=L(x+3)+r''+L.\alpha'+3L.\alpha''=L(x+3)+r'''$$
&c. &c. &c.

If, in the value of $L.\alpha''\cdots n$ (Art. 6), we put successively 0, 1, 2 for n, we have

$$L.\alpha=L\left(\frac{x}{x-1}\right);\quad L.\alpha'=L\left\{\frac{(x-1)(x+1)}{x^2}\right\};\quad L.\alpha''=L\left(\frac{(x+2)x^3}{(x-1)(x+1)^3}\right);$$

or

$$L.\alpha=\ 2M\left\{\frac{1}{2x-1}+\frac{1}{3}\left(\frac{1}{2x-1}\right)^3+\frac{1}{5}\left(\frac{1}{2x-1}\right)^5+\right\}$$
$$L.\alpha'=-2M\left\{\frac{1}{2x^2-1}+\frac{1}{3}\left(\frac{1}{2x^2-1}\right)^3+\frac{1}{5}\left(\frac{1}{2x^2-1}\right)^5+\right\}$$
$$L.\alpha''=\ 2M\left\{\frac{2x+1}{2x^4+4x^3-2x-1}+\frac{1}{3}\left(\frac{2x+1}{2x^4+4x^3-2x-1}\right)^3+\right\}$$

These are the most converging values, I shall show presently how to expand them into series of monomials.

10. If the intended number of decimal places should require $L.\alpha'''$ also to be retained, make, first

$$L.\alpha+L.\alpha'=r$$
$$L.\alpha+2L.\alpha'+L.\alpha''=r+L.\alpha'+L.\alpha''=r'$$
$$L.\alpha+3L.\alpha'+3L.\alpha''+L.\alpha'''=r'+L.\alpha'+2L.\alpha''+L.\alpha'''=r''$$
$$L.\alpha+4L.\alpha'+6L.\alpha''+4L.\alpha'''=r''+L.\alpha'+3L.\alpha''+3L.\alpha'''=r'''$$

. . . .

$$L.\alpha+nL.\alpha'+\frac{n(n-1)}{1.2}L.\alpha''+\frac{n(n-1)(n-2)}{1.2.3}L.\alpha'''=r''\cdots^{(n-2)}+$$
$$L\alpha'+(n-1)L.\alpha''+\frac{(n-1)(n-2)}{1.2}L.\alpha'''=r''\cdot\cdot^{(n-1)}$$

Next make

$$L.\alpha'+L.\alpha''=r_{(1)}$$
$$L.\alpha'+2L.\alpha''+L.\alpha'''=r_{(1)}+L.\alpha''+L.\alpha'''=r'_{(1)}$$
$$L.\alpha'+3L.\alpha''+3L.\alpha'''=r'_{(1)}+L\ \alpha''+2L\alpha'''=r''_{(1)}$$

. . . .

$$L.\alpha'+(n-1)L.\alpha''+\frac{(n-1)\,(n-2)}{1.2}L.\alpha'''=\underset{(1)}{r}^{''\ldots(n-3)}+L.\alpha''+$$
$$(n-2)L.\alpha'''\quad=\underset{(1)}{r}^{''\ldots(n-2)}$$

If $L.\alpha''''$ had been used, we must have made $L.\alpha''+L.\alpha'''$ $=\underset{(2)}{r}$, and have proceeded as before.

The substitutions above being made equations (e) become

$$L.x=L(x-1)+L.\alpha$$
$$(Lx+1)=L.x+L\alpha+L\alpha' \qquad =L.x+r=L.x+r$$
$$L(x+2)=L(x+1)+r+L.\alpha'+L.\alpha'' \qquad =L(x+1)+r'=L(x+1)+r+\underset{(1)}{r}$$
$$L(x+3)=L(x+2)+r'+\underset{(1)}{r}+L.\alpha''+L.\alpha'''=L(x+2)+r''=L(x+2)+r'+\underset{(1)}{r'}$$
$$L(x+4)=L(x+3)+r''+\underset{(1)}{r'}+L.\alpha''+2L.\alpha'''=L(x+3)+r'''=L(x+3)+r''+\underset{(1)}{r''}$$
$$L(x+5)=L(x+4)+r'''+\underset{(1)}{r''}+L.\alpha''+3L.\alpha'''=L(x+4)+r''''=L(x+4)+r'''+\underset{(1)}{r'''}$$

&c. &c.

Where it is plain that each logarithm is found by four additions $\underset{(1)}{r'}$, $\underset{(1)}{r''}$, $\underset{(1)}{r'''}$, &c. being got by two each.

Prop. V.

11. *To construct a Table of Logarithms by means of interpolation from the converging expressions* L . s, L . s', L . s'', &c.

If we consider the formation of equations (d), we easily perceive that the terms of $L(x+n)$ observe the same law, with respect to those of $L(x+n-1)$, which we observed in equations (c); we have therefore by a similar elimination,

$$\begin{aligned}
L.x &= L(x-2)+L.s \\
L(x+1) &= L(x-1)+L.s+L.s' \\
L(x+2) &= L.x+L.s+2L.s'+L.s'' \\
L(x+3) &= L(x+1)+L.s+3L.s'+3L.s''+L.s''' \\
&\cdots\cdots\cdots
\end{aligned}$$

$$L(x+n)=L(x+n-2)+L.s+nL.s'+\tfrac{n(n-1)}{1.2}L.s''+$$

which in order that the logarithms may be got from one another by addition, must be transformed as in the last proposition by the assumption of ρ, ρ', ρ'', &c.; $\rho_{(1)}$, $\rho'_{(1)}$, $\rho''_{(1)}$, &c.; $\rho_{(2)}$, $\rho'_{(2)}$, $\rho''_{(2)}$ &c.: thus if the case only requires us to use $L.s$, $L.s'$, and $L.s''$, make

$$\begin{aligned}
&L.s+L.s'=\rho \\
&L.s+2L.s'+L.s'' = \rho+L.s'+L.s'' = \rho' \\
&L.s+3L.s'+3L.s'' = \rho'+L.s'+2L.s'' = \rho'' \\
&\quad\text{\&c.} \qquad\qquad \text{\&c.} \qquad\qquad \text{\&c.}
\end{aligned}$$

by substituting which our equations become

$$\begin{aligned}
L.x &= L(x-2)+L.s \\
L(x+1) &= L(x-1)+L.s+L.s' = L(x-1)+\rho \\
L(x+2) &= L.x+\rho+L.s'+L.s'' = L.x+\rho' \\
L(x+3) &= L(x+1)+\rho'+L.s'+2L.s'' = L(x+1)+\rho'' \\
&\text{\&c.} \qquad\qquad \text{\&c.} \qquad\qquad \text{\&c.}
\end{aligned}$$

If now we put successively 0, 1, 2 for n in the value of $L.s''^{\cdots\cdots n}$, given in Art. 7, we find

$$L.s=L\left(\frac{x}{x-2}\right);\ L.s'=L\left(\frac{(x+1)(x-2)}{x(x-1)}\right);\ L.s''=L\left(\frac{(x+2)(x-1)^2}{(x+1)^2(x-2)}\right);\ \text{or}$$

$$L.s=2M\left\{\frac{1}{x-1}+\frac{1}{3}\left(\frac{1}{x-1}\right)^3+\frac{1}{5}\left(\frac{1}{x-1}\right)^5+\text{\&c.}\right\}$$

$$L.s' = -2M\left\{\frac{1}{x^2-x-1}+\frac{1}{3}\left(\frac{1}{x^2-x-1}\right)^3+\frac{1}{5}\left(\frac{1}{x^2-x-1}\right)^5+ \&c.\right.$$

$$L.s'' = \quad 2M\left\{\frac{2}{x^3-3x}+\frac{1}{3}\left(\frac{2}{x^3-3x}\right)^3+\frac{1}{5}\left(\frac{2}{x^3-3x}\right)^5+\&c.\right\}$$

(BORDA's Series.)

PROP. VI.

12. *To expand* $L.\alpha''^{\cdots n}$ *and* $L.s''^{\cdots n}$ *into series of monomials of the form* $\frac{A}{x^r}$.

$$L(x+n) = L.x + M\left\{\frac{n}{x}-\frac{n^2}{2x^2}+\frac{n^3}{3x^3}-\ldots\ldots\ldots\ldots\pm\frac{n^r}{rx^r}\mp\right\}$$

$$-\frac{n+1}{1}L(x+n-1) = -\frac{n+1}{1}L.x + M\left\{-\frac{n+1}{1}.\frac{n-1}{x}+\frac{n+1}{1}.\frac{(n-1)^2}{2x^2}\right.$$

$$\left.-\ldots\mp\frac{n+1}{1}.\frac{(n-1)^r}{rx^r}\pm\right\}$$

$$\frac{(n+1)n}{1.2}L(x+n-2) = \frac{(n+1)n}{1.2}L.x + M\left\{\frac{(n+1)n}{1.2}.\frac{n-2}{x}-\frac{(n+1)n}{1.2}.\frac{(n-2)^2}{2x^2}\right.$$

$$\left.+\ldots\pm\frac{(n+1)n}{1.2}.\frac{(n-2)^r}{rx^r}\mp\right\}$$

&c. &c.

These added together will give $L.\alpha''^{\cdots n}$. It is easy to see that $L.x$ will disappear, because its coefficient $=(1-1)^{n+1}$; we have then, putting Σ to represent the sum of the terms formed by the different values of r,

$$L.\alpha^{\cdots n} = \pm M.\Sigma\left\{\frac{n^r-\frac{n+1}{1}(n-1)^r+\frac{(n+1)n}{1.2}(n-2)^r-\frac{(n+1)n(n-1)}{1.2.3}(n-3)^r+.}{rx^r}\right\}$$

where for r we are to take every whole number from one upwards; thus

$$L.\alpha = \quad M\left\{\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{3x^3}+\ldots\ldots\ldots\ldots+\frac{1}{rx^r}+\right\}$$

$$L.\alpha' = -M\left\{\frac{1}{x^2}+\frac{1}{2x^4}+\frac{1}{3x^6}+\&c.\right\}$$

&c. &c.

Hence $L.\alpha'$ equals double the sum of the second, fourth, &c.

terms of $L.\alpha$, with the sign changed. $L.s''^{\cdots n}$ is expanded, by means of the logarithmic series, in a similar manner.

Prop. VII.

13. *To calculate a Table of Logarithmic sines, or cosines.*

It is quite evident, that, if, in the fractional products, in Propositions II, and III, instead of x, $x-1$, $x-2$, &c. we had used successively cos. x, cos. $(x-u)$, cos. $(x-2u)$, &c. or sin. x, sin. $(x-u$ sin. $(x-2u)$, &c., the reasonings made use of would have been equally applicable; and that the whole methods given in Proposition III, IV, V, including the general expressions for $L.\alpha''^{\cdots n}$, $Ls''^{\cdots n}$, (but not the expansions of the said expressions) will hold good here, after we have made the above mentioned substitutions. Thus if it is L. cos. x which we are calculating, we shall have

$$L.\alpha = L\left(\frac{\cos. x}{\cos. (x-u)}\right); \; L.\alpha' = L.\left(\frac{\cos. (x-u) \cos. (x+u)}{\cos.^2 x}\right); \; L.\alpha'' = L\left(\frac{\cos. (x+2u) \cos.^3 x}{\cos. (x-u) \cos.^3 (x+u)}\right)$$

These are the logarithms of numbers converging continually towards unity, and must be found by the form for $L.\left(\frac{a}{b}\right)$.